Area functions, formant frequencies, and spatial phase

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October 2, 2019

Abstract

This manuscript is the final one of a series of three manuscripts exploring relations between area functions and formants. We summarize the results of the first two and provide some context within the previous work in speech acoustics in terms of electrical analogues. The notion of a local, or short region of a tube is central to the considerations of the first manuscript, where we studied how short constrictions can limit formant frequency increase, or substantially decrease formant frequencies. We review the non-local constrictions of the second manuscript, and how they can substantially raise, say, the second formant frequency. If such constrictions are gradually reduced in axial length then we start to revert back to the behavior examined in the first manuscript.

During the exploration of formant behavior with changes in constriction length, the concept of spatial phase arises. This can be defined mathematically and computed for approximations to smooth area functions. Spatial phase is central to relating formant frequencies to area function in both the forward and inverse problems of speech acoustics. The computation of phase can be performed in various ways, and we begin the exploration of these computations in this concluding manuscript of the series.

Introduction

This is the final manuscript in a series of three manuscripts that focus on the relationship between area functions and formants (McGowan, 2019a, 2019b). One of the two main purposes of this manuscript is to provide a larger context for the results of the first two manuscripts. The other main purpose is to explain the importance and computation of something called spatial phase $\Theta^{spatial}$, which was introduced in McGowan (2019b).

All of the work in this series of manuscripts is based on a mechanical view of acoustics. That is, we use the continuum mechanics of air flow with the conservation equations and results from thermodynamics to derive equations that describe acoustic phenomena (McGowan, 2018). We begin with the *acoustic approximation*, which considers the linearized equations for mass conservation and momentum conservation, along with the constancy of entropy. For one-dimensional air motion in a tube of cross-section A, along the tube axis with coordinate x, mass and momentum conservation are (McGowan, 2018; pp. 27-33),

$$\frac{\Delta_t p(x,t)}{\Delta t} = -\frac{\rho_0 c_0^2}{A}\frac{\Delta_x Q(x,t)}{\Delta x}$$
$$\frac{\Delta_t Q(x,t)}{\Delta t} = -\frac{A}{\rho_0}\frac{\Delta_x p(x,t)}{\Delta x} \tag{1}$$

where t is time, p is the pressure perturbation, Q is the perturbation volume velocity, ρ_0 is the rest density of air, and c_0 is the adiabatic speed of sound. For functions of t and x, say $f(t,x)$, $\Delta_t f(t,x) = f(t+\Delta t/2, x) - f(t-\Delta t/2, x)$ and $\Delta_x f(t,x) = f(t, x+\Delta x/2) - f(t, x-\Delta x/2)$. Wave equations for p and Q can easily be derived from the expressions in Equation (1) (McGowan, 2018; pp. 38-40). For instance, for one-dimensional motion,

$$0 = \frac{\Delta_x^2 p(x,t)}{(\Delta x)^2} - \frac{1}{c_0^2}\frac{\Delta_t^2 p(x,t)}{(\Delta t)^2} \tag{2}$$

This is the *wave equation* for p in a tube of constant cross-sectional area A.[1]

[1] Equation (2) is a discretized form of the wave equation. The wave equation results when $\Delta x \to 0$ and $\Delta t \to 0$ and we obtain a *differential equation*. Differential equations are used later in the manuscript.

This means that we can expect wave-like behavior for p and Q, as well as for particle velocity $u = Q/A$. For finite-length tubes standing waves and resonances at particular frequencies Fm, or circular frequencies $\omega_m = (2\pi)Fm$ for $1 \leq m < \infty$, arise. For a tube of constant cross-sectional area A and neglecting losses, resonances are instantiated as these standing waves for p and Q that are are sinusoidal in both space and in time for single frequency disturbances (McGowan, 2018).

Interesting aspects of speech acoustics emerge when cross-sectional area A is no longer a constant, but is a function of position along the tube axis. For this reason we would like to know the relationship between $A = A(x)$ and the acoustic motion in the tube. One of the ways to characterize this acoustic motion is to consider formant frequencies Fm, $1 \leq m < \infty$ or the corresponding circular frequencies ω_m.

Fant employed *distributed* electrical elements in electrical analogue mathematical models for acoustic propagation in the vocal tract (Fant, 1960; p. 28). The elements are *capacitive, inductive*, and *resistive elements*, expressed in terms of per-unit-distance along the tube axis. These mathematical models were also implemented as physical electrical models (Stevens, Kasowski, & Fant, 1953). In fact, as far as the acoustic approximation holds, the mechanical and electrical approaches should provide identical results, whatever $A(x)$. Thus, we see wave-like behavior in the voltage and current in such analogue models.

Area functions can be configured in such a way so as to create special approximations to the relations in Equation (1) in particular regions of the tube. In these situations, while the wave equation remains valid, it is not the best way to understand the physics of the air flow. This is discussed below in relation to a review of the findings in McGowan (2019a,b). Below we relate some of the findings of McGowan (2019a,b) for non-constant area functions to some of the terminology used in electrical analogue modeling, where distributed electrical elements are not the best way to understand the behavior of the voltage and current in the special regions of the tube acoustics.

The concept of spatial phase $\Theta^{spatial}(\omega, x)$ arose in McGowan (2019b), and we pursue this further because of its importance and utility for tubes with variable cross-sectional area. It helps to relate the ares function $A(x)$ and the formant frequencies Fm, $1 \leq m < \infty$, so it can be used in both the *forward problem* and the *inverse problem*. In this manuscript, the forward problem is that of finding mappings from area function to formant frequencies, and the inverse problem is that of finding mappings from formant frequencies to area function.

The first approximation to an area function of a tube considers many sub-tubes of constant cross-sectional area. Thus, we approximate a variable area $A(x)$ as a series of steps with sub-tubes of constant cross-sectional area connecting the steps. In this situation, there are many abrupt changes of cross-sectional area, which necessitates abrupt changes in $\Theta^{spatial}$ at the junctions between sub-tubes. After the required computations of $\Theta^{spatial}$ in this approximation are examined, we go on to discuss the application of the resulting $\Theta^{spatial}$ in both the forward and inverse problems.

We continue the research into $\Theta^{spatial}$ by deriving differential equations for spatial phase and pressure amplitude for gradually varying area functions. These differential equations are restricted in their application: they provide the solution to the Webster horn equation.

We speculate on how spatial phase is calculated in general. If we retain the many constant area sub-tubes in the calculation of spatial phase, we may want a different model to calculate added mass in a variable area tube, such as conical horns. Future research would explore ways of incorporating added mass into the calculation of spatial phase.

The latter part of this manuscript is unapologetically more mathematical than the previous two manuscripts. In other words we need "to slip into something more comfortable", after some basics of spatial phase are discussed. Derivatives, integrals, and a couple of differential equations are used in the latter part of the manuscript. This is unavoidable in order to make progress at this point.