Acoustics of Speech Production

by R. S. McGowan

November 3, 2017

for Winifred and Rebecca

Table of Contents

Preface

Acoustics is the study of sound. Acoustics, as a part of mathematical physics, is a theory of small disturbances in an otherwise quiescent medium, like air. This means that we take the conservation equations describing air motion, and linearize them. There are other assumptions made in the *acoustic approximation*, but thinking of acoustics in air as linearized fluid mechanics is not far off the mark. While general mechanics of air in the human respiratory system is complex, and often non-acoustic, we can usually make the acoustic approximation in most of the supraglottal vocal tract during speech production. When we cannot, the approximation usually breaks down in localized regions, such as at the glottis itself, or in other highly constricted regions. These regions often act as sources of sound for the regions of the vocal tract where air movement is well described by the acoustic approximation.

We make an important distinction immediately regarding measurable quantities and the various approximations that can be used to describe air motion. This distinction has become clouded when air motion described in the acoustic approximation has been related to the motion described by alternative approximations to the conservation equations, such as the Bernoulli equation. These relations have often been made in an ad hoc manner that seem to make it necessary refer to such things as "acoustic pressure" or "Bernoulli pressure". When considering the physics of air, we understand air has certain physical, measurable properties, such as density, pressure, particle velocity, volume velocity, and so on. These quantities are defined in such a way that they do not depend on the particular theoretical approximation used to describe their changes in space and time. For instance, there is no such thing as "acoustic pressure" or "Bernoulli pressure", but there are instances and regions where the behavior of pressure is best described using the acoustic approximation, and others when its behavior is best described by Bernoulli's equation. The motion of air does not accommodate itself according to the theoretical or mathematical approximations that are intended to describe the air's motion.

There were several motivations that I had to write this book. I believe, that to make progress in understanding sounds that are propagated in the atmosphere as a result of speech, the physics of acoustics must be understood by, at least, some researchers. Further, previous books on speech acoustics, such as those of Fant (1960), Flanagan (1965), and Stevens (1998) were written using electrical analogues. Electrical analogues are fine for many calculations, but they assume that the linearized acoustic approximation is valid. It is important to examine acoustics in the broader context of fluid mechanics to understand the use of acoustics for calculation and its connection to fluid motions in the vocal tract that do not conform to the acoustic approximation.

Other motivation came from questions that I have received from linguists, psychologists, and speech clinicians regarding acoustics. One of these questions regarded filter banks used to synthesize speech. If I remember correctly, the question was how is it that frequencies of a source, $\mathcal{F}$, such as the harmonics of the voice source, drive the filters in a filter bank, but do not follow the relation $\mathcal{F} = c/\lambda$, where c is the speed of sound and λ is the wavelength of the resonance represented by a particular filter in the filter bank? Questions like this made it clear that the way acoustics is traditionally presented to speech researchers is as a computational tool. The physics has often been lost in our understanding when computing outputs from inputs with transfer functions and filter banks. I hold to the idea that "The purpose of theory is understanding, not calculation."

This is a book about settled science, as one of my colleagues has said. Almost all the the material in this book comes from nineteenth and early twentieth century physics. The newest topic that is in the book, other than the two research topics in Chapter 14, is in Chapter 13, where we discuss how air motion that is not described by the acoustic approximation can serve as a source of acoustic energy. This is the study of aeroacoustics, and its modern beginnings came just after the first half of the twentieth century with the publication of Lighthill's paper on jet noise in 1952.

The book is divided into four parts. Chapters 1 through 7 provide a thorough explanation of acoustic motion in a tube of finite length and constant cross-sectional area with a moving piston at one end. Some discussion of another type of source is also included. In the first part of the book, almost all of the results are presented in what we consider the time domain. This part lays out the fundamental mathematical physics of the situation, so that further developments in variable-area tubes in Chapters 8 through 12 become more a matter of computation. However, we never switch completely to just computation of physical quantities, but continue to provide physical understanding with the derivation of equations. Much of the second part of the book involves frequency domain considerations. The third part of the book, which is in Chapter 13, is an introduction to fluid mechanics that does not follow the acoustic approximation. This leads to a short study of aeroacoustics and the way that the fluid motions that do not satisfy the acoustic approximation can provide acoustic energy to the vocal tract. Finally, Chapter 14 outlines two research projects that have not hitherto been published. The first involves a hypothesized relation between tongue surface curvature constraints applied to young children and their resulting acoustic output during speech. The second work is a proposed model for vocal fold vibration that uses some of the ideas introduced in Chapter 13. It is intended as a replacement for lumped element models, such as the two-mass model. Biomechanical parameters can be more easily related to the proposed model than to lumped element models. This final topic uses advanced mathematical tools.

This book is targeted at people interested in the physical aspects of acoustics that arise during the act of speaking. This includes people involved directly with the science of speech production, but without advanced mathematical knowledge, and those in the physical sciences with some advanced mathematical knowledge. [Here, we consider anything more than a smattering of calculus to be advanced knowledge of mathematics.] It is written to be accessible by those who are not trained in mathematics, while not holding back on presenting important physical ideas that are expressed mathematically. There are many simulations presented in plots and with numerical values to aid understanding. The book is something of a narrative about the physical acoustics encountered during speech production. It is intended to be read in sequence. This is not a text book, or even a reference book, in the traditional sense. For one thing, there are no problems for the student. However, if the reader wishes a more thorough understanding, he or she could derive some of the mathematical expressions presented in the book.

The reader should just forge ahead if he or she finds that some mathematical expressions are too difficult to understand. There is always a discussion or simulation that should still be illuminating. We would also recommend that readers use software to simulate some of the results, and turn to references for more in-depth discussion of topics of interest.

Equations are numbered separately in each chapter. Further, Chapters 1, 2, and 8 have short mathematical appendices with equations numbered as, for example (A1.2), where A1 refers to the Appendix to Chapter 1, and 2 refers to the equation number in that appendix. In some derivations, equation numbers are written in square brackets next to the steps of the derivation, so the reader can justify the step. We do this only for the first few uses of equations in the appendices.

There are numerous people to thank. There are friends, those who I have worked with in speech production research, and those across the United States who have either hosted me or encouraged me on my cross-country treks to teach in the American southwest. The number of people is too great to mention them by name. I want to thank my colleagues at Imperial Valley College, and my friends on both sides of the American-Mexican border in the college's region. All of these people helped me realize that a pedagogical vocation may not be out of reach for me.

Finally, I thank the people who read earlier versions of this book: Michael Howe, Lynn McGowan, Philip Rubin, and Reiner Wilhelms-Tricarico. They helped to improve this book immensely and are encouraging friends as well.

R. S. McGowan
Lexington, Massachusetts

References

Fant, G. (1960). *Acoustic Theory of Speech Production.* Mouton, The Hague.

Flanagan, J.L. (1965). *Speech Analysis, Synthesis, and Perception.* Springer-Verlag, New York.

Lighthill, M.J. (1952). On sound generated aerodynamically, Part 1: General Theory. *Proc. Roy. Soc. A.* **211**, 564-587.

Stevens, K.N. (1998). *Acoustic Phonetics.* MIT Press, Cambridge, MA.

Chapter 1

Air: The Acoustic Material in the Vocal Tract and Some Basic Physics

Introduction

Acoustics is the study of small amplitude, unsteady, or time-varying, disturbances to physical materials. These materials provide restoring forces to the parts of the material that are moved away from their undisturbed rest position. These restoring forces tend to return the parts that are moved away from rest position to their position of undisturbed rest. An example of a restoring force is the force exerted by a spring that returns to its rest length after it is compressed or stretched and released. This definition is too broad though, because it applies to vibrations, as well as to what we generally think of as acoustics. Often it is difficult to distinguish vibration from acoustics, so we immediately restrict our considerations to air, as this is the principal medium for vocal tract acoustics. For a classification of the general subject matter covered by the term acoustics, see Pierce (1989).

Properties of air

Air is the gas that surrounds the earth. We have an intuitive idea of a gas like air, but here we characterize air as physicists do. Air is a part of the class of fluids that also includes liquid water. While both gases and liquids are fluids, gases have the property that they can be compressed much more readily than liquids, and gases expand to fill the available space. Fluids gain potential energy when they are compressed because they possess restoring forces, which means that acoustic motion can occur in these media. However, neither bodies of liquids nor gases retain their shapes when external forces are applied (Batchelor 1967).

Let's consider air under so-called normal conditions. For us, normal conditions correspond to the state of atmospheric air at sea level at 15° Celsius, or 15° C. In discussing volumes of air, the cubic centimeter, cm^3, which can be denoted cc, is a common unit. Initially, one may consider this to be a cube measuring 1 cm on each edge. In our examination of air, the smallest length scales of interest are very much larger than the inter-molecular distances, i.e. the mean distance between molecules in air. For example, 10^{-3} cm, or 1 one-thousandth of a cm, is much larger than inter-molecular distances (Batchelor 1967). This corresponds to a volume of 10^{-9} cm^3 or 10^{-9} cc, or 1 billionth of a cc: a very small volume of air on a terrestrial or even human scale. Batchelor (1967) writes that there are about 3×10^{10}, or 30 billion, molecules of air in that apparently small volume at normal pressure and temperature. Small volumes of air have something called mass, m, associated with them; mass is a measure of the amount of "matter" in the small volume. A particular unit of mass is the gram, which is denoted g.

We have been introducing measurement units, such as volume, cm^3, and mass g. Of course, the volume cm^3 is based on a length measure cm. We are using the system of measurement that has cm as length, g as mass, and seconds, s, for time. We usually work in the metric system, and within that system, we usually work in what is known as c-g-s units for centimeters, grams, and seconds.

One-dimensional kinematics and dynamics of a point mass

Before continuing the discussion of air, we review some concepts associated with the movement of small bodies of mass. These small bodies are often idealized as mathematical points that possess mass. Thus, we consider some of the *kinematics* and *dynamics* of masses that are reduced to mathematical points in the next sections. Kinematics is the study of motion without regard to the causes of the motion, while dynamics includes the causes of motion, such as force.

Kinematics

All motions of a point mass considered here are one dimensional along the x-axis, say. The point mass is sometimes referred to as a body and is given the coordinate x that is a function of the time, t, i.e. $x = x(t)$. This body could represent one of the small volumes of air that was introduced above. The $x(t)$ coordinate provides information on the body's distance from the origin along the x axis, $|x(t)|$, as well as information on whether it is to the right or left of the origin. That is, the distance of the body from the origin is $+x(t)$ when the body is to the right of the origin, when $x(t) > 0$, and the distance to the origin is $-x(t)$ when the body is to the left of the origin, when $x(t) < 0$. Also, motion can be in the positive x-direction (rightward motion), or the negative x-direction (leftward motion). Figure 1 shows an example plot of the position of a body as a function of time. For $t < 1$ s the body has positions $x(t) < 0$. It moves to the right so that its positions become positive, i.e. $x(t) > 0$ after $t = 1$ s for some time. At some time between $t = 1$ s and $t = 2$ s the body starts to move to the left, so that by $t = 2$ s it is heading back into negative-x territory. Between $t = 2$ s and $t = 3$ s, the body starts to move to the right again, so that after $t = 3$ s, $x(t) > 0$.

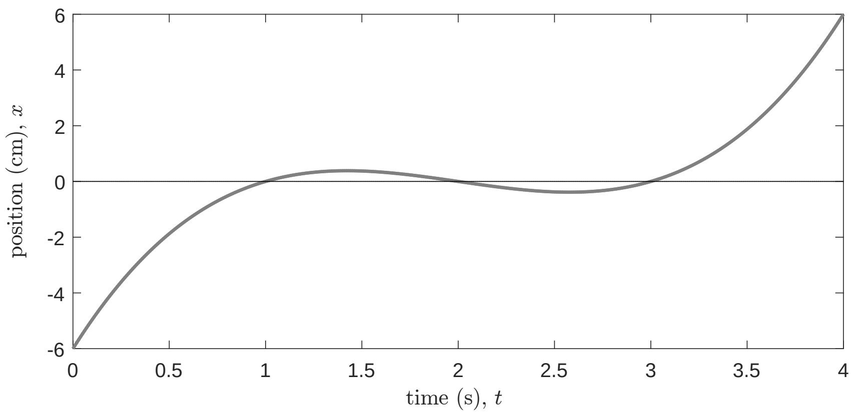

Figure 1. Body, or mass, position versus time

If the point mass is moving rightward, its position has a positive time rate-of-change, or positive *velocity*, where velocity $v(t)$ at time t can be defined approximately as the ratio of the change in position over the change in time over an interval $t + \Delta t/2 > t > t - \Delta t/2$, with a small $\Delta t > 0$,

$$
\begin{aligned}
v(t) &\approx \frac{x(t + \Delta t/2) - x(t - \Delta t/2)}{\Delta t} \\
&\equiv \frac{\Delta_t \; x(t)}{\Delta t} \qquad (1)
\end{aligned}
$$

where the approximation improves as the time interval Δt decreases. The sign $\equiv$ means that we are defining Δ_t as

$$\Delta_t f(t) \equiv f(t + \Delta t/2) - f(t - \Delta t/2) \qquad (2)$$

This is known as a *first difference* of the function $f(t)$. Two properties of the first difference are examined in the Appendix to Chapter 1.

Velocity $v(t)$ is approximately the ratio of the first difference of position, $x(t)$, to the difference in time, t. [We often drop the approximation symbol, "$\approx$", and simply write equal, "$=$".] This is the slope of the line joining points $(t - \Delta t/2, x(t - \Delta t/2))$ and $(t + \Delta t/2, x(t + \Delta t/2))$. This line is called the secant line, and Figure 2 shows the secant line for the particular time $t = 1$ and position $x(t = 1) = 0$ of Figure 1.

Slopes at any time, t, can be calculated. These slopes, are, in turn, a function of time, t. $v(t) > 0$ when the body is moving to the right at time t, and $v(t) < 0$ when the body is moving to the left. These ideas are illustrated in Figure 3 for the position, $x(t)$, shown in Figure 1. Figure 3 shows the plot of the velocity, $v(t)$, below the plot of the position as a function of time, $x(t)$. The lower plot is simply a plot of the slopes of the small line segments around each point of the plot of position versus time. Samples of these line segments on the position versus time plot and their

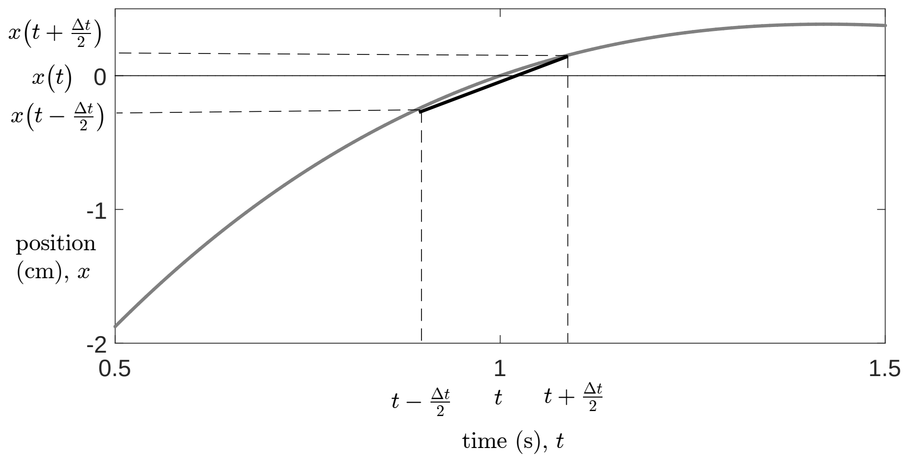

FIGURE 2. The curve, $x(t)$, and the secant line at $t = 1$ and $x(t = 1) = 0$

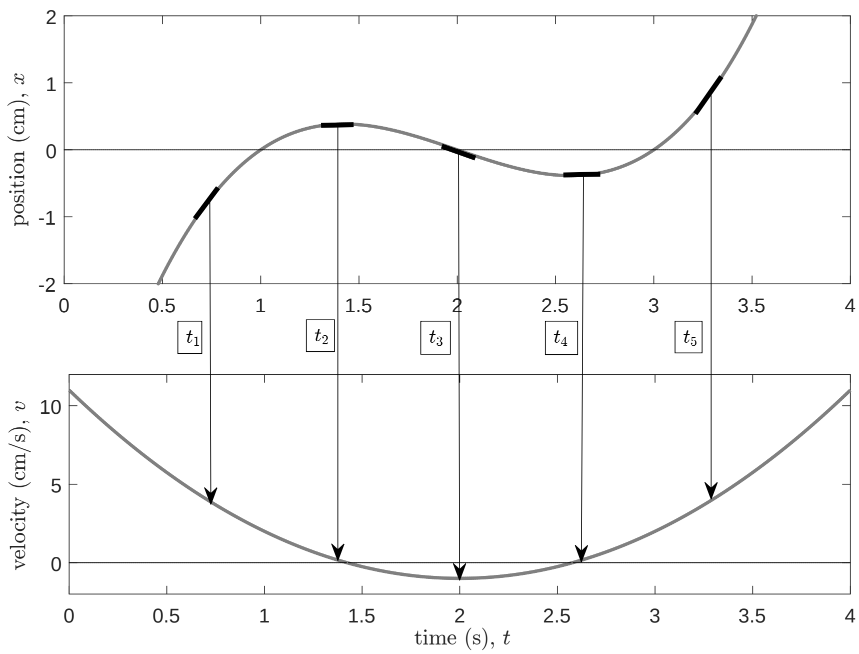

FIGURE 3. Mass position versus time and velocity versus time

corresponding slopes on the velocity versus time graph are indicated by arrows. We see that the velocity is positive through the time t_2, while the position goes from negative to positive before this time. The body stops at the time t_2 with zero velocity, and then starts to move left, as indicated by the fact that the velocity is negative, and the position of the body goes from positive to negative at the time t_3. At the time t_4, the body has stopped and then begins moving to the right again, because the velocity is positive after this time. It continues to move in this direction indefinitely. The reader should note that there can be any combination of positive and negative signs for the pair $x(t)$ and $v(t)$. For example, a body can be to the left of the origin, yet moving rightward, with $x(t) < 0$ and $v(t) > 0$, as in Figure 3. The absolute value of the velocity, or velocity magnitude, is called the *speed*.

Acceleration $a(t)$ is the time rate-of-change of velocity,

$$a(t) \approx \frac{\Delta_t v(t)}{\Delta t} = \frac{v(t + \Delta t/2) - v(t - \Delta t/2)}{\Delta t} \tag{3}$$

where the approximation improves as the time interval $\Delta t > 0$ decreases. It follows from Equations (1) and (3) that,

$$\begin{aligned} a(t) &= \frac{\frac{x(t+\Delta t)-x(t)}{\Delta t} - \frac{x(t)-x(t-\Delta t)}{\Delta t}}{\Delta t} \\ &= \frac{\frac{\Delta_t\ x(t+\Delta t/2)}{\Delta t} - \frac{\Delta_t\ x(t-\Delta t/2)}{\Delta t}}{\Delta t} \\ &= \frac{\Delta_t\ x(t + \Delta t/2) - \Delta_t\ x(t - \Delta t/2)}{(\Delta t)^2} \\ &= \frac{\Delta_t\ \big(x(t + \Delta t/2) - x(t - \Delta t/2)\big)}{(\Delta t)^2} \quad \text{[Equation (A1.4)]} \\ &= \frac{\Delta_t\big(\Delta_t\ x(t)\big)}{(\Delta t)^2} \quad \text{[Equation (2)]} \\ &\equiv \frac{\Delta_t^2 x(t)}{(\Delta t)^2} \end{aligned} \tag{4}$$

Acceleration $a(t)$ is the ratio of the first difference of velocity, $v(t)$, to the difference in time, Δt. Equation (4) shows that acceleration is equivalent to the *second difference* in position, $x(t)$, divided by the square of the difference in time, $(\Delta t)^2$. The quantities $x(t), v(t)$, and $a(t)$ are related to one another by the ratios of differences shown in Equations (2) and (4).

Figure 4 shows the relationship between the velocity, $v(t)$, and acceleration, $a(t)$, for the position and velocity of Figures 1 and 3. The acceleration is negative up until the time t_3. From before the time t_1 until the time t_2, the mass' velocity is positive, but it is decreasing in magnitude, or speed. From the time t_2 to the time t_3, the mass is going to the left with an increasing speed, or magnitude of velocity. Both of these intervals provide a negative acceleration. From the time t_3, the acceleration is positive. From this time until the time t_4, the mass velocity is negative, but its speed is decreasing. From the time t_4 onward, the body's velocity is positive and increasing in speed. Both of these intervals provide a positive acceleration.

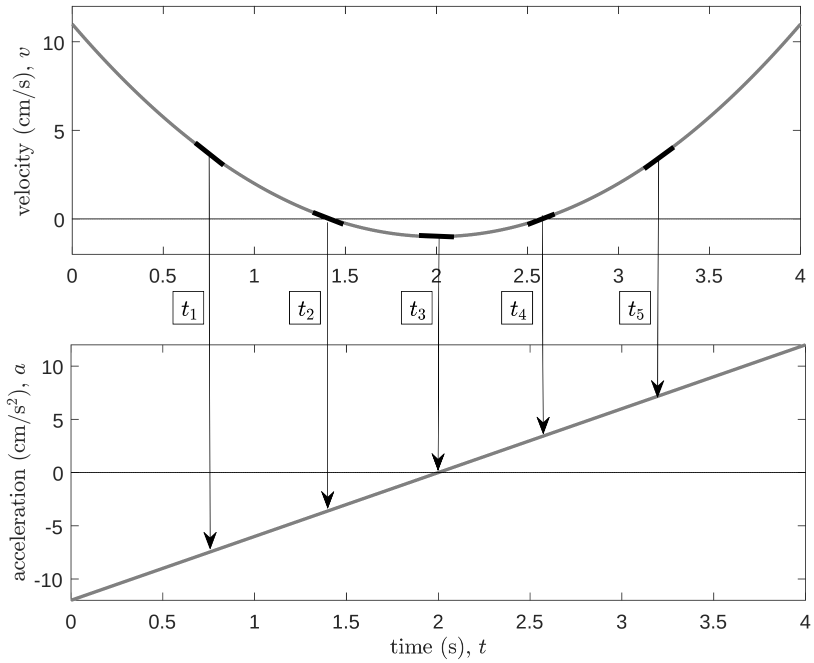

FIGURE 4. Mass velocity versus time and acceleration versus time

Dynamics

The momentum of a body of mass m and velocity v is defined to be mv. Newton's second law of motion states that the time rate-of-change of momentum of a body is equal to the force exerted on the body,

$$F = \frac{\Delta_t(mv)}{\Delta t} = m\frac{\Delta_t v}{\Delta t} = ma \tag{5}$$

Newton's second law relates force on a body of constant mass to acceleration. The constant of proportionality between force F and acceleration a is mass m.

The earth exerts a gravitational force on objects with mass. If we were to throw a ball up from the ground, the ball would accelerate toward the center of the earth, that is, down toward the ground, at approximately constant acceleration. The ball would go up initially, but with decreasing speed. It would eventually start to fall back toward the ground with increasing speed. The constant acceleration due to gravity is g, which is known as the *acceleration of gravity.* The numerical value of g is approximately 980 cm/s^2 near the earth's surface. The force exerted on an object by the earth is given the symbol W, where

$$W = m\mathsf{g} \tag{6}$$

When an object is at rest on the surface of the earth, the surface exerts a force of magnitude W upward on the object, and the object exerts a force of the same magnitude downward on the earth, as required by Newton's third law. W is known as the *weight* of the object.

In daily life we tend to mix mass and weight. In some countries for instance, meat is often labeled according to pounds and grams. Pound (lb) is a force, or weight, unit in the English system, and the gram is a mass unit in the metric system. The weight or force measure in the c-g-s system is a dyne, where dyne = g cm/s^2. So, for instance, 2.5 g of meat actually weighs 2450 dyne.

Thermodynamic properties of air

We return to the discussion of air using what are termed *thermodynamic* properties. We often refer to things like air pressure and temperature in our everyday life, as in the weather forecast. Pressure and temperature are examples of thermodynamic quantities. Thermodynamic quantities characterize small volumes of air that are composed of many molecules, such as the 10^{-9} cm^3 volume with upwards of 30 billion air molecules. Statistical analyses can be performed with this many molecules, and thermodynamic quantities can be related to averages of dynamic quantities of the molecules, as well as other expected values. For instance, temperature is related to average molecular kinetic energy. One of the triumphs of theoretical physics is statistical mechanics, which takes statistical properties of a large number of molecules in a substance like air and relates the bulk thermodynamic properties to the translational motions and other degrees-of-freedom of molecular motion.

Density is a thermodynamic quantity that is relatively easy to define. It is simply the amount of mass in a unit of volume, and it is denoted by ρ. Under the normal conditions defined above, air has a density of 1.225×10^{-3} g/cm^3 = 0.001225 g/cm^3 (Batchelor 1967). We write $\rho_{atm} = 1.225 \times 10^{-3}$ g/cm^3. This compares to liquid water, which has a density very close to 1 g/cm^3 under the same pressure and temperature conditions. It is important to keep in mind that air is very much less dense than water; very roughly, air has one-thousandth the density of water. Because the human body is largely composed of water, this gives us a good approximation of the ratio of the density of air in the vocal tract to the density of surrounding tissue.

Pressure is a thermodynamic quantity that can be characterized by considering the dynamics of air molecules. Pressure is denoted p. The intuitive idea of pressure is that of air pushing against the elastic walls of a balloon as it is being blown up. The agent that is blowing the balloon up is imparting a higher pressure to the air inside the balloon than the pressure of air outside the balloon, which is atmospheric. If one evacuates air from a basketball, it crinkles because the interior pressure is below atmospheric pressure. Air moves from regions of high pressure to regions of low pressure, although this is getting into the relation between pressure and motion of bodies of air, so this is ahead of the discussion.

From a mechanical perspective, pressure is a *stress* and has the units of force per unit area, such as dyne/cm^2. For air at rest, it is the only stress and it is termed *normal stress*, because it is directed in the normal direction to any mathematical surface considered to be within the air. Normal means perpendicular here; to be directed normal to a surface is to be directed perpendicular to that surface. If we approximate air using a model of gas molecules analogous to small billiard balls, then the pressure within a small volume is proportional to the average translational energy per unit volume of the molecules within the small volume (Batchelor 1967).

Often, we consider mathematical surfaces surrounding a small volume of air. For air at rest, it does not matter how the surfaces of the volume are defined: a spherical surface, a cubical surface, or any other surface surrounding a small volume. A non-zero pressure p means that there is the *possibility* of force in a direction normal to surfaces. The magnitude of force across the surface of small area A has magnitude pA. If the surface is planar, then force acts in a direction normal to the plane. If the surface is spherical then the force acts along the radius of the sphere. However, the force exerted on the surface by the air inside the volume is exactly counterbalanced by the pressure exerted by the air outside the volume when the volume of air is not moving. Thus, the net force on the surface is zero.

Pressure can be measured with many devices, including water and mercury U-tube manometers, as shown in Figure 5. A U-tube manometer can be made from a glass tube of constant cross-section, A_m, that is bent into a U shape. The U-tube is partially filled with a liquid, water or mercury, and closed at one end, which is evacuated, so that the pressure at the surface of the liquid on that side of the U-tube is approximately zero. The other end is left open to the atmosphere so that the liquid surface on this side of the U-tube experiences the atmospheric pressure, p_{atm}. We take the value of p_{atm} to be its value under normal conditions, 1.013×10^6 dyne/cm^2 (Batchelor 1967). In order for the liquid to remain at rest the net force on the liquid must be zero, so that the gravitational force on the liquid must balance the force supplied by the pressure of the atmosphere. Let h denote the difference in the liquid elevation between the sides of the U-tube. The upward force on the liquid in the open side of the U-tube is equal to the difference in weight of liquid on the closed side and the open side. The mass of the excess liquid on the closed side is $\rho_l A_m h$, where ρ_l is the density of the liquid. Thus, its weight is $W = \rho_l A_m h\mathtt{g}$, where $\mathtt{g}$ is the acceleration of gravity. The force pushing the liquid down on the open side of the U-tube is $p_{atm} A_m$. In order for these forces to balance one

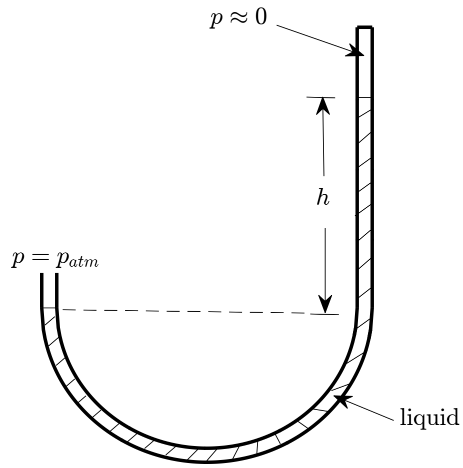

FIGURE 5. U-tube manometer

another, their magnitudes must be equal, so that $p_{atm} = \rho_l \mathsf{g} h$. Thus, for a given liquid, the atmospheric pressure, p_{atm}, is proportional to the height difference in the liquid, h. This is why pressure is often quoted in units of cm H_2O (centimeters of water), or in units of mm Hg (millimeters of mercury).

It is usual to express speech related pressures in terms of centimeters of water, cm H_2O. The pressure is in relation to atmospheric pressure, so that, for instance, 10 cm H_2O denotes a pressure above atmospheric and −10 cm H_2O denotes a pressure below atmospheric. The following is the conversion factor between atmospheres and cm H_2O.

$$1 \text{ atmosphere} = 1033 \text{ cm H}_2\text{O} \approx 10^3 \text{ cm H}_2\text{O} \tag{7}$$

Thus, 10 cm H_2O is about 1% of atmospheric pressure. Another useful relation is,

$$1 \text{ atmosphere} = 1.013 \times 10^6 \text{ dyne/cm}^2 \approx 10^6 \text{ dyne/cm}^2 \tag{8}$$

In the discussion of pressure, we have assumed that the small volumes are, as a whole, at rest. Note that the velocity of a small volume of air with its 30 billion molecules is distinct from the velocities of the molecules that compose the volume. Velocities of molecules are generally random in direction and the distribution of their magnitudes, or speeds, is determined by the temperature of the air. If the volume of air has a velocity as a whole, then the velocity of the molecules is the sum of the random velocities and the velocity of the volume. When small volumes of air are in motion, the definition of pressure becomes a little more problematic, because the normal stresses are not the same in all directions. Thus, pressure is simply defined as the mean (over the three Cartesian directions) of the normal stresses (Batchelor 1967).

There are thermodynamic quantities of air that we don't talk about in our everyday lives, such as entropy. However, there is one fact to remember: we need only two thermodynamic quantities to completely characterize the equilibrium thermodynamics of a given volume of air. That is, we can choose any two quantities, such as temperature and pressure, and calculate the other quantities, such as density and entropy from an *equation of state*. We do not discuss equations of state in general, as this would take us too far from the topic at hand. A particular relation between pressure and density under restrictive assumptions is provided below.

The concept of entropy is difficult, and it is best understood with statistical mechanics. However, it is fortunate that we are able to make a large simplification because of the following assumption. We assume that entropy is uniform through the region of air that is under investigation, and, further, the entropy does not change with time. [For an initially homogeneous medium this would mean that changes in energy of small air masses due to heat conduction and friction are negligible. There are places where this can be violated in speech acoustics, and we attempt to point those out later in Chapter 11. The regions where there is change in entropy are usually confined to be very close to solid surfaces.] With a completely constant entropy, we can find the equilibrium value for any of temperature, pressure, or density as a function of any one of these quantities. Let's suppose we take pressure p as the independent thermodynamic variable, so that both temperature and density ρ are considered functions of pressure.

In our discussion of the relation between thermodynamic variables, we state a result for small thermodynamic disturbances in lieu of a full equation of state. Let p_0 be the pressure of undisturbed air (i.e. atmospheric) and ρ_0 the undisturbed density of air. Further, let $\delta p = p - p_0$ and $\delta\rho = \rho - \rho_0$, be the pressure and density of a small disturbance, respectively. In other words, we assume that $|\delta p|/p_0$ and $|\delta\rho|/\rho_0$ are both much smaller than one, or that $|\delta p|/p_0 \ll 1$ and $|\delta\rho|/\rho_0 \ll 1$. The disturbances are related by,

$$\frac{\delta\rho}{\delta p} = c_0^{-2} \quad \text{or} \quad \frac{\delta p}{\delta\rho} = c_0^2 \tag{9}$$

One can work out that the dimensions of c_0 are those of a velocity, e.g. cm/s. It is called the *adiabatic speed of sound*, and we take the value of $c_0 = 34,100$ cm/s under normal conditions, for which $p_0 = p_{atm}$ and $\rho_0 = \rho_{atm}$ (Batchelor 1967). The term adiabatic comes from the constancy of entropy, and we see later why it is the speed of sound.

In order to reference physical quantities pertaining to normal conditions, we write them in equation form.

$$\begin{aligned} \rho_{atm} &= 1.225 \times 10^{-3}\ \text{g/cm}^3 \\ p_{atm} &= 1.013 \times 10^6\ \text{dyne/cm}^2 = 1033\ \text{cm H}_2\text{O} \\ c_0 &= 34,100\ \text{cm/s} \end{aligned} \tag{10}$$

Geometry

The air continuum

In mathematical theory air is considered to be a *continuum*: it is a mathematical three-dimensional space that is filled with a mathematical "material" that has the physical properties of air. Each point in the mathematical space can be conceived as representing a small volume of air, with its thermodynamic and kinematic properties (Landau & Liftshitz 1959). The molecular level of physical description is completely ignored here.

The continuum framework permits each point to be treated as a *fluid particle*, or an *air particle*. Air particles have associated thermodynamic properties, such as pressure, density, temperature, and entropy, as well as the kinematic quantities of position, velocity and acceleration. Obviously, this is an abstraction, because a point without extent is now representing a small volume. The distinction between physical air and the mathematical space that represents a continuum with each point having the properties of air may seem ponderous. However, we are presenting a physico-mathematical theory, so clarity is important.

Eulerian and Lagrangian frames-of-reference

In our discussion of the kinematics of point masses, we followed an individual point mass through a one-dimensional space. This frame-of-reference is called the *Lagrangian* frame-of-reference. It is the frame-of-reference in which we describe the physics of a finite number of bodies, whether point masses or extended masses. We may be familiar with this frame-of-reference from previous experience with introductory academic physics. On the other hand, our default frame-of-reference for the kinematics and dynamics of air is called the *Eulerian* frame-of-reference (Landau & Lifshitz 1959), for which we fix points in space and consider what happens as air particles pass through these points. We do not follow the individual air particles. We often shorten Eulerian or Lagrangian frame-of-reference to Eulerian or Lagrangian frame.

Figure 6 illustrates the difference between the two frames of reference. The top two panels show the Eulerian frames at times t_1 and $t_2 > t_1$. The bottom two panels show the Lagrangian frame at the same two times. The box at $x = x_1$ is the object of investigation in the Eulerian frame, with different air particles A and B at that location at the two different times. The object of investigation in the Lagrangian frame is the air particle A,

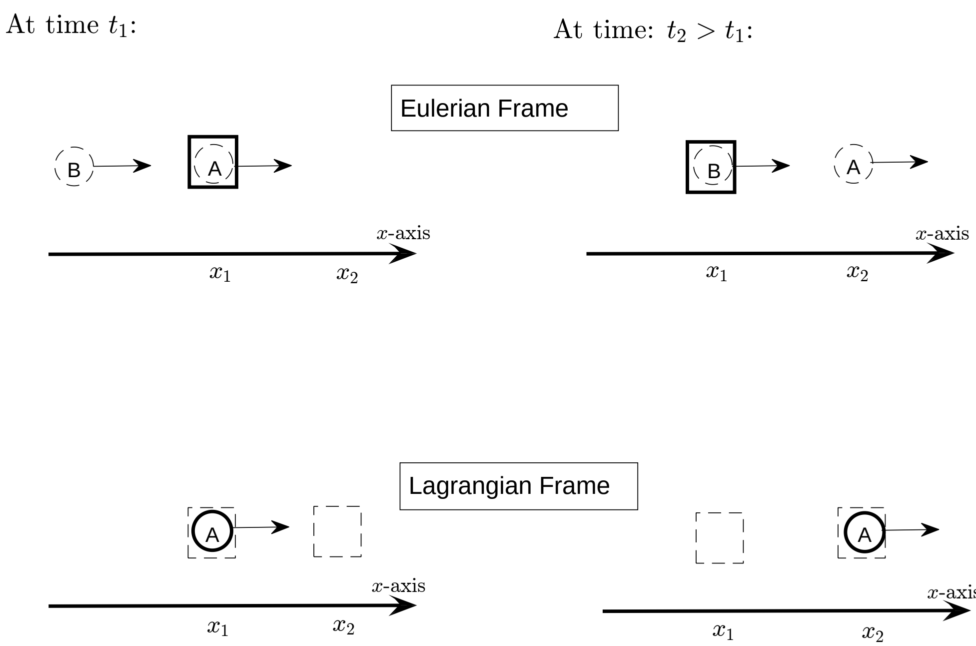

FIGURE 6. Eulerian and Lagrangian frames

which is in two different boxes, at x_1 and x_2 at the two different times. While the laws of motion are the same in either frame of reference, the way that they are written depends on our frame of reference. Before continuing into the mechanics of air in the Eulerian frame in the next chapter, we examine mass-spring systems in the Lagrangian frame.

Mass-spring systems

A simple mass-spring system

Consider a massless spring that can be compressed or extended in the x-direction. We suppose that the spring's left end is attached to a infinitely massive wall. Supposes that an external agent either stretches or compresses the spring to length L, from a rest length of L_0. The rest length is the spring's *equilibrium* state. [We often use the term agent to denote some person or machine that manipulates the system of interest, but itself is not a part of that system.] We assume that the spring obeys Hooke's law

$$F_{res} = -\kappa(L - L_0) \tag{11}$$

The $(L - L_0)$ is the difference between the spring length, L, and the undisturbed, or rest length, L_0. F_{res} is the restoring force of the spring, with $\kappa > 0$ known as the *spring constant*, or *spring stiffness.* If the spring is compressed by an agent at the right end of the spring, so that $(L - L_0) < 0$,

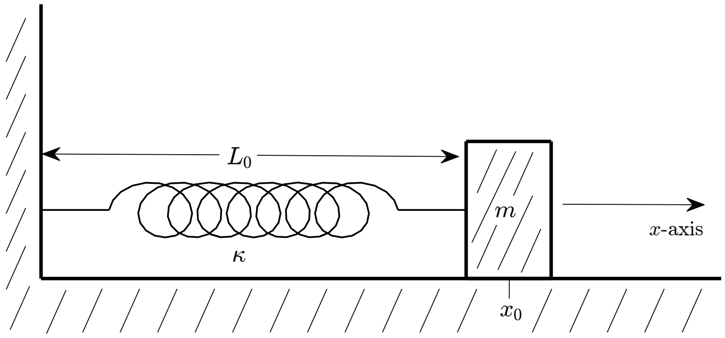

FIGURE 7. A simple mass-spring system

then the spring exerts a force against the agent in the positive x-direction, that is, toward its equilibrium. If the spring is stretched, then $(L - L_0) > 0$, and the restoring force acts against the agent to pull in the negative x-direction, again, toward the equilibrium length L_0.

Now we attach a mass m to the right end of the spring so that the mass moves along a frictionless surface as the spring compresses and stretches The position of the mass when the spring has length L is denoted x_m, and the particular position of the mass when the spring is at its rest length, L_0, is written x_0. Thus, $x_m - x_0 = L - L_0$. We let $X_m = x_m - x_0$, which is referred to as *displacement position*, or, simply, *displacement* here. When there is no external agent present, the restoring force of the spring, F_{res}, acts on the mass. The entire configuration, the mass-spring system, is shown in Figure 7.

If there is no external agent acting on the mass-spring system, then by Newton's second law, expressed in Equation (5),

$$ma = F_{res} = -\kappa(L - L_0) = -\kappa X_m(t),$$

(12)
$$\begin{aligned} \text{where } a &= \frac{\Delta_t^2 x_m(t)}{(\Delta t)^2} \\ &= \frac{\Delta_t^2 x_m(t)}{(\Delta t)^2} - \frac{\Delta_t^2 x_0}{(\Delta t)^2} \quad \text{[Equation A1.2]} \\ &= \frac{\Delta_t^2\big(x_m(t) - x_0\big)}{(\Delta t)^2} \quad \text{[Equation A1.4]} \\ &= \frac{\Delta_t^2 X_m(t)}{(\Delta t)^2} \end{aligned}$$

where a is the acceleration of the mass. Rewriting Equation (12),

(13)
$$m\frac{\Delta_t^2 X_m}{(\Delta t)^2} + \kappa X_m(t) = 0$$

Equation (13) is the *equation of motion* for the mass-spring system.

Suppose the mass-spring system is left undisturbed until an external agent agent moves the mass to x_m just before time $t = 0$, and then releases the mass from a state of rest at $t = 0$. This means that the spring is initially compressed or stretched by the external agent. That is, for $X_m(t = 0) \neq 0$ and $\Delta_t X_m(t = 0)/(\Delta t) = 0$. These are *initial conditions*.

The limiting case of $\Delta t \to 0$ is considered in the following. We happen to know the analytic form of the solution to Equation (13) in this limit.

(14) $$X_m(t) = x_m(t) - x_0 = \mathcal{A}\cos(\omega_0 t + \theta), \quad \text{for } t > 0$$

where $\mathcal{A}$ and θ are determined by the initial conditions. In our case $\mathcal{A} = X_m(t = 0) = x_m(t = 0) - x_0$ and $\theta = 0$. Here we have defined,

(15) $$\omega_0 = \sqrt{\frac{\kappa}{m}}$$

ω_0 has the dimensions of circular frequency, s^{-1}, and is the *natural circular frequency* and $\mathcal{F}_0 = \omega_0/(2\pi)$ is the *natural frequency* for the mass-spring system. The dimensions of $\mathcal{F}_0$ are also s^{-1}. The values of $\mathcal{F}_0$ are usually stated in Hz (Hertz). The reason for the term “frequency” is made clear in the following paragraph.

Equation (14) specifies *simple harmonic motion*, because its solution is written as a *circular function* of time. In this instance the circular function is cos, or cosine. The properties of circular functions, cos, and sin are reviewed in Chapter 3. For the present purposes we simply note the properties in Figure 8, where the cos function oscillates between -1 and 1 symmetrically about the vertical position 0. Figure 8 shows the solution in Equation (14) when the initial displacement $X_m(t = 0) = -1$ and the natural frequency $\mathcal{F}_0 = 100\ Hz$. The position displacement, $X_m(t)$, repeats itself every $T = 0.01\ s$. So we see that $\mathcal{F}_0 = 1/T$, and this is the reason for the term natural frequency. According to Equation (15), natural circular frequency, and, hence, natural frequency, can be increased by either increasing the spring constant, or stiffness, κ, or by decreasing the mass, m. The times $t \leq 0$ are not shown in Figure 8, because the system is not described by the equation of motion, Equation (13), for $t \leq 0$.

Figure 9 shows the displacement position, $X_m(t)$, as a function of time along with the corresponding velocity, $v(t)$. Figure 10 shows the velocity, $v(t)$, and corresponding acceleration, $a(t)$, as functions of time. Consider Figures 9 and 10 at the times corresponding to the arrows. At the time t_1, the mass is at its most negative position, and changing its direction of motion from negative to positive, and its velocity is zero. At the same time, the acceleration is at its maximum positive value. This follows from the fact that acceleration is proportional to force, which is in the opposite direction to the displacement of the mass from equilibrium. Note that the change in direction corresponds to maximum acceleration, even as the

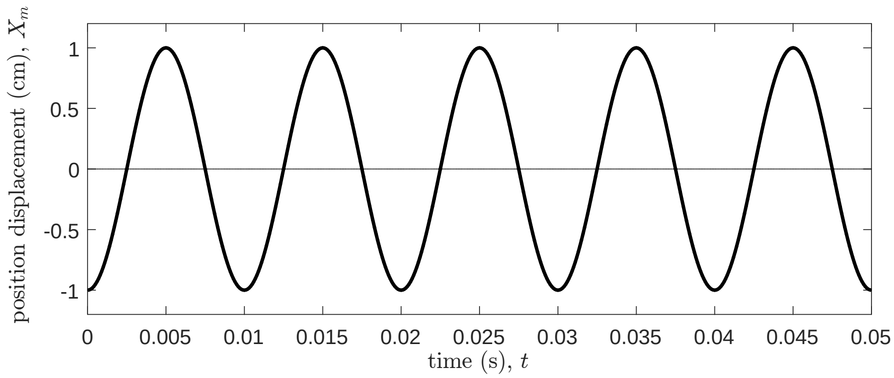

FIGURE 8. Mass position displacement versus time

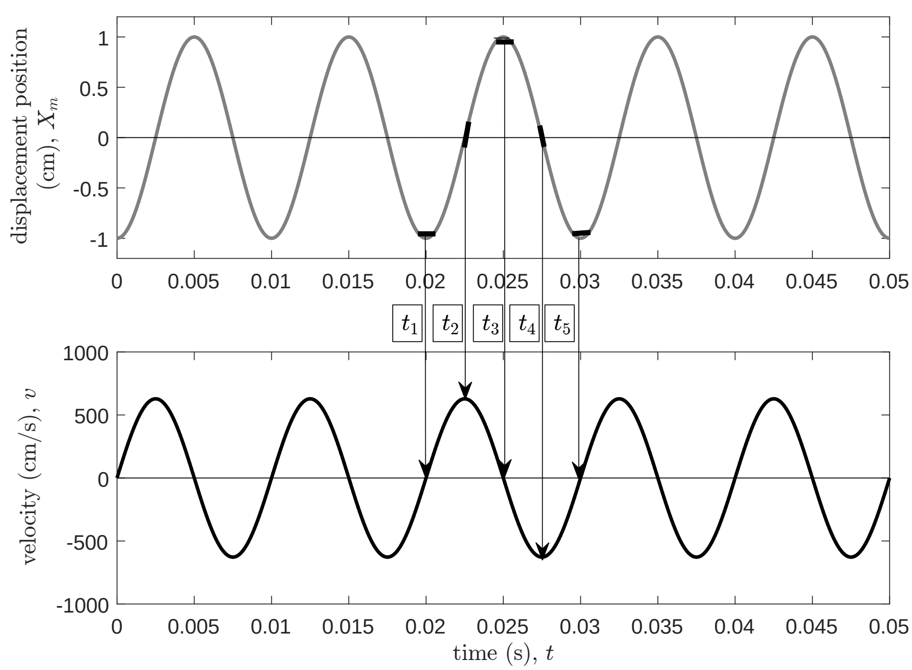

FIGURE 9. Mass position displacement versus time and mass velocity versus time

velocity is zero. At the time t_2, the mass passes through equilibrium in the positive direction, where it has its maximum velocity. The acceleration is zero because the mass is at its equilibrium position at that instant. At time t_3, the situation is that the mass has reached its maximum positive position, so that the acceleration is maximally negative, and the velocity is zero. We leave the reader to discover the properties of motion at times t_4

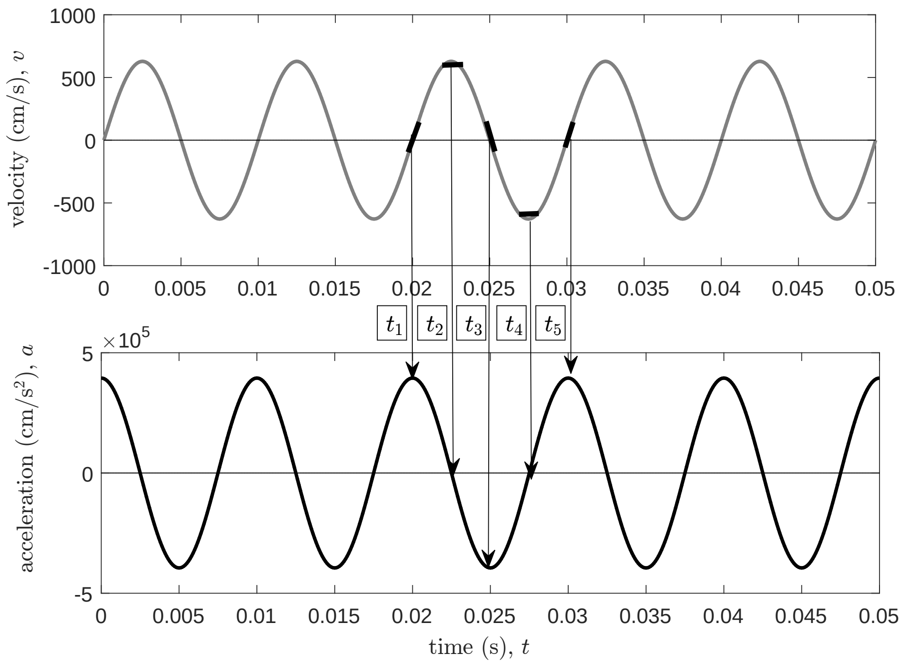

FIGURE 10. Mass velocity versus time and mass acceleration versus time

and t_5. The plots seem to indicate that both the velocity and acceleration are also circular functions, which is indeed the case.

We have seen that when displacement is small that the speed of the mass is large, and when displacement is large that the speed of the mass is small [see Figure 9]. Kinetic energy is $E^{kin} = mv^2/2$. Therefore, the kinetic energy of the mass is inversely relate to the stretch or compression of the spring. We can even think that the mass' kinetic energy "gives" itself up to stretch or compression of the spring, and, conversely, the stretch or compression of the spring "give" themselves up to kinetic energy of the mass. If we define potential energy as $E^{pot} = \kappa(L - L_0)^2/2 = \kappa X_m^2/2$, then it is seen that there is an indefinite oscillation between the kinetic energy of the mass and the potential energy of the spring. This is an important characteristic of oscillating systems, such as the simple-mass spring system.

A mass-spring system involving air

Let's examine a small container of air with a constriction tube that contains a movable solid mass, m, in one of the sides of the container. The constriction tube has cross-sectional area A_c. Figure 11 shows the configuration. We assume that the mass provides an air-tight seal, and that it is able to move freely without friction in the x-direction. Under normal

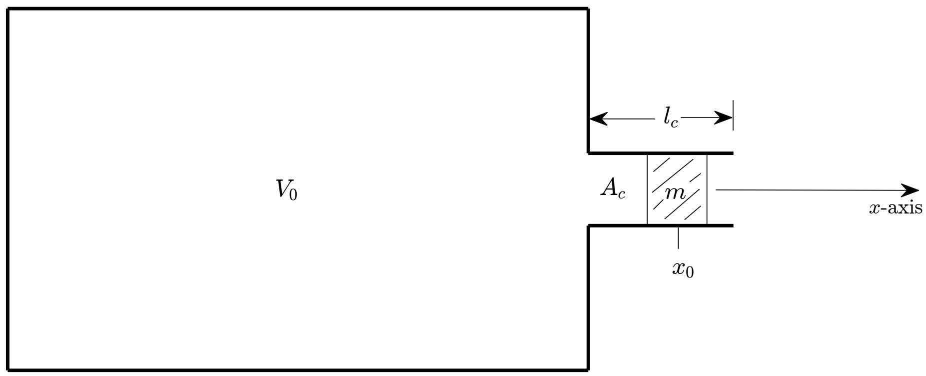

FIGURE 11. A mass-spring system for air

atmospheric conditions, the mass is in equilibrium at $x_m = x_0$, and the volume of air in the container behind the mass is V_0. In this equilibrium the air both inside and outside the container are under normal atmospheric conditions.

Suppose that the mass is moved to position x_m, so that $X_m = x_m - x_0$. The volume available to the air in the container is changed to $V_0 + \Delta V = V_0 + A_c X_m$, where $\Delta V = A_c X_m$. Let ρ and p denote the density and pressure of air in the container. Because there is no air taken away or added to the container, the relationship between changes of air density, ρ, and changes in volume, V, is an inverse relation,

$$\frac{\Delta\rho}{\rho_0} = -\frac{\Delta V}{V_0} \tag{16}$$

Under certain conditions that are outlined later in the book, we can assume that changes in density and pressure in the container are approximately uniform and established approximately instantaneously. Thus, while density and pressure can be functions of time, their values at any time are uniform throughout the container. It is best to have a container with small linear dimensions for this assumption to hold true.

If we assume that the relationship between changes in density and changes in pressure given in Equation (9) holds for small, but finite changes in density and pressure then,

$$\begin{aligned} \Delta p &= c_0^2 \Delta\rho \\ &= -\frac{\rho_0 c_0^2}{V_0}\Delta V \\ &= -\frac{\rho_0 c_0^2}{V_0} A_c X_m \end{aligned} \tag{17}$$

This follows from Equation (16). The net force on the mass in the x-direction is $F = (p - p_0)A_c = \Delta p A_c$, where p_0 is atmospheric pressure, p_{atm}. It follows that,

$$F = -\frac{\rho_0 c_0^2 A_c^2}{V_0} X_m \tag{18}$$

This force is a restoring force, because it acts in a direction opposite to X_m. The constant $\rho_0 c_0^2 A_c^2/V_0$ is equivalent to the spring constant that appears in Hooke's law, in Equation (11). In fact, the equation of motion for the mass, m, is given by Equation (13) with

$$\kappa = \rho_0 c_0^2 \frac{A_c^2}{V_0} \tag{19}$$

and the mass executes simple harmonic motion. The air in the large volume, V_0, supplies the restoring force for simple harmonic motion.

It should be of no surprise that the mass in the constriction tube could be supplied by air, instead of a solid mass. Suppose we remove the solid mass so that only air is in the constriction tube. If the constriction tube has length l_c, then the mass of air in the sleeve, m_c, is

$$m_c = \rho_0 l_c A_c \tag{20}$$

[We see in Chapter 12 that we should use a length slightly larger than the physical length of the tube l_c.] With air as the mass in the constriction tube we obtain the natural frequency from Equations (15), (19), and (20).

$$\mathcal{F}_H = \frac{1}{2\pi}\sqrt{\frac{\kappa}{m_c}} = \frac{1}{2\pi} c_0 \sqrt{\frac{A_c}{l_c V_0}} \tag{21}$$

This is a rough estimate of the natural frequency, $\mathcal{F}_H$, of a *Helmholtz resonator*. It takes some time to see that this is a valid model for the acoustics of the vocal tract in special geometries. We have derived this only to illustrate that air can behave like a simple harmonic oscillator. Unlike the simple mass-spring system, the air provides for both the mass and the spring in the Helmholtz resonator.

Conclusion

The quantity in the expression for spring constant, κ, in Equation (19) has a factor that is intrinsic to air, namely $\rho_0 c_0^2$. This can be thought of as the inherent stiffness of air, where something that is stiff is not easily compressed. Note that water also has springiness, but it is much stiffer than air: not only is water almost 10^3 times denser than air, but the c_0^2 is also larger in water. In air $c_0 \approx 3.41 \times 10^4$ cm/s and in water $c_0 \approx 1.45 \times 10^5$ cm/s (Batchelor 1967). This means that the ratio of $\rho_0 c_0^2$ in water to the same quantity in air is approximately 1.8×10^4, meaning that water is much stiffer than air. One could expect that acoustic communication in the oceans to differ substantially from that in the atmosphere.

We relate $\rho_0 c_0^2$ to fractional changes in density of air and small change in pressure. From Equation (9),

$$\frac{\delta\rho}{\rho_0} = \frac{1}{\rho_0 c_0^2}\delta p \equiv D\ \delta p \tag{22}$$

where $D = 1/(\rho_0 c_0^2)$ is known as the *distensibility* of air (Lighthill 1978). This quantity is the constant of proportionality between changes in pressure, δp, and fractional changes in density. The smaller D the more pressure it takes to attain a given fractional density change. Thus, again, we see that $\rho_0 c_0^2 = 1/D$ is something of a spring constant for air. This spring provides the restoring force that allows for what we usually associate with acoustic air motion.

In Chapters 2 and 3, we show that the springiness and mass-like like qualities of air results in *wave motion*. This takes us well beyond the special case of the Helmholtz resonator, where air is configured to act as a mass-spring system. On the other hand, the wave motion can be represented by systems that are mathematically mass-spring like in the case of wave motion in a tube of finite length, such as the vocal tract. Therefore, after further study of simple mass-spring systems in Chapter 4, we pursue this mass-spring like mathematical representation in Chapter 5.

References

Batchelor, G.K. (1967). *An Introduction to Fluid Dynamics*. Cambridge University Press: Cambridge, England. (pp. 1-4, 6, 38-40, 141, 594, 596)

Landau, L.D. & Lifshitz, E.M. (1959). *Fluid Mechanics*. Pergamon Press: Oxford, England. (pp. 1-5)

Lighthill, M.J. (1978). *Waves in Fluids*. Cambridge University Press: Cambridge, England. (p. 93)

Pierce, A.D. (1989). *Acoustics: An Introduction to its Physical Principles and Applications*. Acoustical Society of America: Woodbury, NY. (pp. 3, 28-36)

Appendix to Chapter 1

Let A be a number. This number does not depend on any variable. Therefore, from the definition of first difference in Equation (2),

$$\Delta_t A = A - A = 0 \tag{A1.1}$$

Therefore,

$$\frac{\Delta_t A}{\Delta t} = 0 \tag{A1.2}$$

for $\Delta t > 0$.

We now prove that Δ_t and $\Delta_t/\Delta t$ are *linear* operators. Let $f(t)$ and $g(t)$ be functions of t, including the possibility of a constant. Let A and B be numbers.

$$\begin{aligned}\Delta_t\big(A\cdot f(t) + B\cdot g(t)\big) &= \big(A\cdot f(t+\Delta t/2) + B\cdot g(t+\Delta t/2)\big) - \\ &\qquad \big(A\cdot f(t-\Delta t/2) + B\cdot g(t-\Delta t/2)\big) \\ &= \big(A\cdot f(t+\Delta t/2) - A\cdot f(t-\Delta t/2)\big) + \\ &\qquad \big(B\cdot g(t+\Delta t/2) - B\cdot g(t-\Delta t/2)\big) \\ &= A\cdot\big(f(t+\Delta t/2) - f(t-\Delta t/2)\big) + \\ &\qquad B\cdot\big(g(t+\Delta t/2) - g(t-\Delta t/2)\big) \\ &= A\cdot\Delta_t f(t) + B\cdot\Delta_t g(t)\end{aligned} \tag{A1.3}$$

It follows that,

$$\frac{\Delta_t\big(A\cdot f(t) + B\cdot g(t)\big)}{\Delta t} = A\cdot\frac{\Delta_t f(t)}{\Delta t} + B\cdot\frac{\Delta_t g(t)}{\Delta t} \tag{A1.4}$$